

CONVOLUTION PRESERVES PARTIAL SYNCHRONICITY OF LOG-CONCAVE SEQUENCES

H. HU, DAVID G.L. WANG^{†‡}, F. ZHAO, AND T.Y. ZHAO

ABSTRACT. In a recent proof of the log-concavity of genus polynomials of some families of graphs, Gross et al. defined the weakly synchronicity relation between log-concave sequences, and conjectured that the convolution operation by any log-concave sequence preserves weakly synchronicity. We disprove it by providing a counterexample. Furthermore, we find the so-called partial synchronicity relation between log-concave sequences, which is (i) weaker than the synchronicity, (ii) stronger than the weakly synchronicity, and (iii) preserved by the convolution operation.

1. INTRODUCTION

The log-concavity of sequences of nonnegative numbers has been paid extensive and intensive attention during the past thirty years, see Stanley [10] and Brenti [3, 4]. In the late 1980s, Gross et al. [6] posed the LCGD conjecture that the genus polynomial of every graph is log-concave, which firstly connected the log-concavity of sequences with topological graph theory, or more precisely, with the surface embedding of graphs. For survey books of topological graph theory, see [1, 7]. In the recent work [5], Gross et al. established a criterion determining the log-concavity of sum of products of log-concave polynomials. With aid of the criterion, they confirmed the LCGD conjecture for several families of graphs generated by vertex- or edge-amalgamations, including the graphs called iterated 4-wheels.

The criterion is considered to have its own interest, since it deals with the intrinsic arithmetic relations between log-concave polynomials. See [2, 8, 9] for related papers. The idea of the criterion consists of three key parts, the synchronicity, the radio-dominance, and the lexicographicity. It is the synchronicity part, which originally arises from common facts observed from topological embeddings of graphs into surfaces, starts the whole development of the new log-concave results.

Though the synchronicity relation is sufficient to judge the log-concavity of positive linear combination of log-concave polynomials, Gross et al. managed to weaken it to certain weakly synchronicity relation. The first power of such a weaker relation was

Key words and phrases. Log-concavity.

supposed to be preserved by sequence convolution, which was posed as the following conjecture; see [5, Conjecture 2.13].

Conjecture 1.1. Let A, B, C be three log-concave nonnegative sequences without internal zeros. If $A \sim_w B$, then the convolution sequences $A * C$ and $B * C$ are weakly synchronized.

We disprove Conjecture 1.1 by providing an explicit counterexample. This example leads us to find a relation in Definition 3.2, called partial synchronicity, between log-concave sequences, to achieve the original motivation. Namely, the partial synchronicity relation is (i) weaker than synchronicity, (ii) stronger than the weakly synchronicity, and (iii) preserved by the convolution operation. See Theorems 3.4 and 3.7.

2. PRELIMINARY AND THE COUNTEREXAMPLE

All sequences concerned in the paper consists of nonnegative numbers. For any finite sequence $A = (a_k)_{k=0}^n$ of nonnegative numbers, we identify the sequence A with the infinite sequence $(a'_k)_{k \in \mathbb{Z}}$, where $a'_k = a_k$ for $0 \leq k \leq n$, and $a'_k = 0$ otherwise. Under this convenience, one may denote $A = (a_k)$ for simplicity. We write uA to denote the scalar multiple sequence (ua_k) , for any constant $u \geq 0$. Let $B = (b_k)$ be another sequence of nonnegative numbers. Then the notation $A + B$ stands for the sequence $(a_k + b_k)$.

We call the first positive term of the sequence A the *head* of A , and call the last positive term the *tail* of A . In other words, the term a_h is said to be the head of A if $a_{h-1} = 0 < a_h$. In this case, we call the integer h the *head index* of A , denoted $h(A) = h$. Similarly, one may define the *tail index*, denoted as $t(A)$. It is clear that $h(A) \leq t(A)$. Without loss of generality, we suppose that $h(A) \geq 0$ for all sequences concerned in this paper.

The sequence A is said to be *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for all integers k . It is said to *have no internal zeros* if for any integers $i < j$ such that $a_i a_j > 0$, one has $\prod_{k=i}^j a_k > 0$. Denote by \mathcal{L} the set of log-concave sequences without internal zeros. We call the sequence consisting of only zeros the *zero sequence*, denoted (0) . Denote

$$\mathcal{L}^* = \mathcal{L} \setminus \{(0)\}.$$

Definition 2.1. Let $A = (a_k) \in \mathcal{L}$ and $B = (b_k) \in \mathcal{L}$. We say that the sequences A and B are *synchronized*, denoted as $A \sim B$, if

$$a_{k-1}b_{k+1} \leq a_k b_k \quad \text{and} \quad a_{k+1}b_{k-1} \leq a_k b_k$$

for all k .

It is obvious that scalar multiplications preserve synchronicity. Moreover, the synchronicity relation is reflexive, symmetric and non-transitive; see [5].

Definition 2.2. Let $A = (a_k) \in \mathcal{L}$ and $B = (b_k) \in \mathcal{L}$. We say that the sequences A and B are *weakly synchronized*, denoted $A \sim_w B$, if

$$(2.1) \quad a_{k-1}b_{k+1} + a_{k+1}b_{k-1} \leq 2a_kb_k,$$

for all k .

For example, consider the sequences $A = (1, 3, 5)$ and $B = (1, 4, 13)$. It is easy to verify that $A \sim_w B$ and $A \not\sim B$.

Recall that if $A = (a_k)_{k=0}^m$ and $B = (b_h)_{h=0}^n$, the convolution sequence $A * B$ is defined to be the coefficient sequence of the polynomial product

$$\left(\sum_{i=0}^m a_i x^i \right) \left(\sum_{j=0}^n b_j x^j \right).$$

The next example disproves Conjecture 1.1.

Example 2.3. Let

$$\begin{aligned} A &= (1, 20, 200, 1800), \\ B &= (1, 6, 30, 60), \\ C &= (40, 60, 10, 1). \end{aligned}$$

It is direct to verify $A \sim_w B$ from Definition 2.2, and to compute that

$$\begin{aligned} A * C &= (40, 860, 9210, 84201, 110020, 18200, 1800), \\ B * C &= (40, 300, 1570, 4261, 3906, 630, 60). \end{aligned}$$

Then, for the convolution sequences $A * C$ and $B * C$, Ineq. (2.1) does not hold for $k = 2$:

$$\begin{aligned} &(A * C)_1(B * C)_3 + (A * C)_3(B * C)_1 - 2(A * C)_2(B * C)_2 \\ &= 860 \times 4261 + 84201 \times 300 - 2 \times 9210 \times 1570 \\ &> 0. \end{aligned}$$

3. THE PARTIAL SYNCHRONICITY RELATION

In this section, we introduce the partial synchronicity relation between log-concave sequences, which is expected to serve the original motivation of Gross et al. in [5].

Let $A = (a_k)$ and $B = (b_k)$ be two sequences of numbers. For any integers m and n , we define

$$(3.1) \quad f(A, B; m, n) = a_m b_n + a_n b_m.$$

When there is no confusion, we simply denote

$$f(m, n) = f(A, B; m, n).$$

From Def. (3.1), we see that the function $f(m, n)$ is commutative, namely,

$$(3.2) \quad f(m, n) = f(n, m)$$

for all integers m and n . For further discussion, we need the following lemma.

Lemma 3.1. *Suppose that*

$$(3.3) \quad f(m, n) \geq f(m+1, n-1)$$

for all integers m and n such that $m \geq n$. Then we have

$$(3.4) \quad f(a, b) \geq f(c, d)$$

for any integers a, b, c, d such that

$$(3.5) \quad a + b = c + d, \quad \text{and that}$$

$$(3.6) \quad |a - b| < |c - d|.$$

Proof. Let a, b, c, d be integers satisfying Eq. (3.5) and Ineq. (3.6). In order to show Ineq. (3.4), one may suppose, by the commutativity Eq. (3.2), that $a \geq b$ and $c \geq d$. Then Ineq. (3.6) reduces to

$$(3.7) \quad a - b < c - d.$$

Summing up Eq. (3.5) and Ineq. (3.7), one obtains that

$$(3.8) \quad a < c.$$

Substituting $m = a$ and $n = b$ in the premise Ineq. (3.3), one finds that

$$(3.9) \quad f(a, b) \geq f(a+1, b-1).$$

Since $a \geq b$, we have $a+1 \geq b-1$. Therefore, in Ineq. (3.9), by replacing the number a by $a+1$, and replacing b by $b-1$, we obtain that

$$(3.10) \quad f(a+1, b-1) \geq f(a+2, b-2).$$

The same substitution for Ineq. (3.10) gives that

$$f(a+2, b-2) \geq f(a+3, b-3).$$

Continuing in this way, one finds

$$(3.11) \quad f(a+i-1, b-i+1) \geq f(a+i, b-i)$$

for all positive integers i . Since $a < c$ from Ineq. (3.8), we can sum up Ineq. (3.11) over $i \in \{1, 2, \dots, c-a\}$, which yields that

$$f(a, b) \geq f(c, b-c+a).$$

Hence, we obtain the desired Ineq. (3.4), by noticing $d = b - c + a$ from Eq. (3.5). \square

Definition 3.2. Let $A, B \in \mathcal{L}$. We say that the sequences A and B are *partially synchronized*, denoted by $A \sim_p B$, if Ineq. (3.3), or equivalently,

$$(3.12) \quad a_m b_n + a_n b_m \geq a_{m+1} b_{n-1} + a_{n-1} b_{m+1},$$

holds for all integers m and n such that $m \geq n$.

It is clear that scalar multiplications preserve partial synchronicity. Moreover, the partial synchronicity relation is reflexive, symmetric, and non-transitive. The non-transitivity can be seen from the example

$$A = (1, 2, 3), \quad B = (1, 3, 8), \quad C = (1, 4, 15),$$

where $A \sim_p B$, $B \sim_p C$, and $A \not\sim_p C$. In fact, this above example has been used to exemplify the non-transitivity of the synchronicity relation in [5].

The next proposition helps check quickly the weakly synchronicity of two sequences, which is also of help in the proof of Theorem 3.4.

Proposition 3.3. Let $A, B \in \mathcal{L}^*$. Then $A \sim_p B$ holds iff

$$(i) \quad |h(A) - h(B)| \leq 1;$$

$$(ii) \quad |t(A) - t(B)| \leq 1; \text{ and}$$

$$(iii) \quad \text{Ineq. (3.12) holds for all integers } m \text{ and } n \text{ such that } m \geq n,$$

$$\max\{h(A), h(B)\} \leq m \leq \max\{t(A), t(B)\} - 1,$$

and that

$$\min\{h(A), h(B)\} + 1 \leq n \leq \min\{t(A), t(B)\}.$$

Proof. Let $A = (a_k)$ and $B = (b_k)$ be sequences such that $A, B \in \mathcal{L}^*$.

Necessity. Suppose that $A \sim_p B$, i.e., Ineq. (3.12) holds for all integers m and n such that $m \geq n$.

In order to show (i), one may suppose that $h(A) \leq h(B)$ without loss of generality. Assume that $|h(A) - h(B)| \geq 2$. Take

$$m = h(B) - 1 \quad \text{and} \quad n = h(A) + 1.$$

It follows that $m \geq n$, and that Ineq. (3.12) becomes

$$(3.13) \quad a_{h(B)-1} b_{h(A)+1} + a_{h(A)+1} b_{h(B)-1} \geq a_{h(B)} b_{h(A)} + a_{h(A)} b_{h(B)}.$$

From definition of the head $h(B)$, we have

$$(3.14) \quad b_{h(B)-1} = 0.$$

On the other hand, since $h(B) - h(A) \geq 2$, we have

$$(3.15) \quad b_{h(A)} = 0 \quad \text{and} \quad b_{h(A)+1} = 0.$$

Substituting Eqs. (3.14) and (3.15) into Ineq. (3.13), one obtains

$$(3.16) \quad 0 \geq a_{h(A)} b_{h(B)}.$$

From the definition of the head function h , one sees that $a_{h(A)} > 0$ and $b_{h(B)} > 0$, contradicting Ineq. (3.16).

Condition (ii) can be shown along the same lines. The necessity of (iii) is obvious from the premise $A \sim_p B$.

Sufficiency. For convenience, we denote

$$\begin{aligned} mh &= \min\{h(A), h(B)\}, & mt &= \min\{t(A), t(B)\}, \\ Mh &= \max\{h(A), h(B)\}, & Mt &= \max\{t(A), t(B)\}. \end{aligned}$$

If $m \geq Mt$, then $a_{m+1} = b_{m+1} = 0$, and thus

$$(3.17) \quad f(m+1, n-1) = 0.$$

Since $f(m, n) \geq 0$, Ineq. (3.3) holds. Below we can suppose that

$$(3.18) \quad m \leq Mt - 1.$$

In another case that $n \leq mh$, we have $a_{n-1} = b_{n-1} = 0$. Therefore, we infer Eq. (3.17), which allows us to suppose without loss of generality that

$$(3.19) \quad n \geq mh + 1.$$

In view of (iii), Ineqs. (3.18) and (3.19), it suffices to prove Ineq. (3.3) for all integers m and n such that $m \geq n$, and that either $m \leq Mh - 1$ or $n \geq mt + 1$.

When $m \leq Mh - 1$, by using Ineq. (3.19), one may deduce that

$$Mh - 1 \geq m \geq n \geq mh + 1,$$

contradicting Condition (i), which implies that $Mh - mh \leq 1$. When $n \geq mt + 1$, by using Ineq. (3.18), we can derive that

$$mt + 1 \leq n \leq m \leq Mt + 1,$$

contradicting Condition (ii), which implies that $Mt - mt \leq 1$. This completes the proof. \square

Now we can clarify the relations among the synchronicity, the weak synchronicity, and the partial synchronicity.

Theorem 3.4. *The partial synchronicity relation \sim_p is weaker than synchronicity \sim , and stronger than the weakly synchronicity \sim_w . In other words, any two synchronized log-concave sequences without internal zeros are partially synchronized, and any two partially synchronized log-concave sequences without internal zeros are weakly synchronized.*

Proof. Taking $m = n = k$ in Ineq. (3.12) gives Ineq. (2.1), which implies that the partial synchronicity \sim_p is stronger than weakly synchronicity \sim_w .

Let $A = (a_k) \in \mathcal{L}$ and $B = (b_k) \in \mathcal{L}$. Let $m \geq n$. It suffices to show Ineq. (3.12). By Proposition 3.3 (iii), we can suppose that

$$m \leq \max\{t(A), t(B)\} - 1 \quad \text{and} \quad n \geq \min\{h(A), h(B)\} + 1.$$

Thus we have $a_{m+1}b_{m+1} \neq 0$ and $a_nb_n \neq 0$. Since $A, B \in \mathcal{L}$, neither of the sequences A and B has internal zeros. It follows that

$$\prod_{i=n}^{m+1} (a_i b_i) \neq 0.$$

By dividing Ineq. (3.12) by the factor $a_{m+1}b_{m+1}$, we see that it is equivalent to prove

$$(3.20) \quad \frac{a_m b_n}{a_{m+1} b_{m+1}} + \frac{a_n b_m}{a_{m+1} b_{m+1}} \geq \frac{b_{n-1}}{b_{m+1}} + \frac{a_{n-1}}{a_{m+1}}.$$

Following the notation in [5], we let

$$\alpha_k = \frac{a_k}{a_{k-1}} \quad \text{and} \quad \beta_h = \frac{b_h}{b_{h-1}},$$

when $a_{k-1} \neq 0$ and $b_{h-1} \neq 0$. Then the desired Ineq. (3.20) can be recast as

$$(3.21) \quad \frac{1}{\alpha_{m+1}} \prod_{i=n+1}^{m+1} \frac{1}{\beta_i} + \frac{1}{\beta_{m+1}} \prod_{i=n+1}^{m+1} \frac{1}{\alpha_i} \geq \prod_{i=n}^{m+1} \frac{1}{\beta_i} + \prod_{i=n}^{m+1} \frac{1}{\alpha_i}.$$

Multiplying Ineq. (3.21) by the product $\prod_{i=n}^{m+1} (\alpha_i \beta_i)$, we find to show the following inequality is sufficient:

$$\beta_n \prod_{i=n}^m \alpha_i + \alpha_n \prod_{i=n}^m \beta_i \geq \prod_{i=n}^{m+1} \alpha_i + \prod_{i=n}^{m+1} \beta_i.$$

That is, it suffices to show that

$$(3.22) \quad (\beta_n - \alpha_{m+1}) \prod_{i=n}^m \alpha_i + (\alpha_n - \beta_{m+1}) \prod_{i=n}^m \beta_i \geq 0.$$

On the other hand, the synchronicity relation $A \sim B$ implies that

$$\alpha_n \geq \beta_{n+1} \quad \text{and} \quad \beta_n \geq \alpha_{n+1}.$$

By the log-concavity of the sequence B , the sequence β_k is decreasing. Thus we have

$$(3.23) \quad \alpha_n \geq \beta_{n+1} \geq \beta_{m+1}.$$

For the same reason, we have

$$(3.24) \quad \beta_n \geq \alpha_{m+1}.$$

In view of Ineqs. (3.23) and (3.24), the desired Ineq. (3.22) follows immediately. This completes the proof. \square

Gross et al. [5, Theorems 2.10, 2.11] showed that any collection of pairwise synchronized sequences is closed under linear combinations with nonnegative coefficients, and that the same property holds for the weak synchronicity relation. We show that partial synchronicity behaves in the same manner in Lemma 3.5 and Theorem 3.6.

Lemma 3.5. *Let $A, B \in \mathcal{L}$ such that $A \sim_p B$. Then we have $uA + vB \in \mathcal{L}$ for all nonnegative numbers u and v .*

Proof. Let $A = (a_k)$ and $B = (b_k)$ be log-concave sequences such that $A \sim_p B$. Let $u, v \geq 0$. Since the sequence A is log-concave, we have

$$(3.25) \quad u^2 a_k^2 \geq u^2 a_{k-1} a_{k+1}.$$

For the same reason, the log-concavity of the sequence B implies that

$$(3.26) \quad v^2 b_k^2 \geq v^2 b_{k-1} b_{k+1}.$$

Since $A \sim_p B$, one may take $m = n = k$ in Ineq. (3.12), which yields

$$(3.27) \quad uv(a_k b_k + a_k b_k) \geq uv(a_{k+1} b_{k-1} + a_{k-1} b_{k+1}).$$

Adding Ineqs. (3.25) to (3.27) up, we obtain that

$$(ua_k + vb_k)^2 \geq (ua_{k+1} + vb_{k+1})(ua_{k-1} + vb_{k-1}).$$

In other words, the sequence $uA + vB$ is log-concave. \square

Theorem 3.6. *Suppose that the sequences A_1, A_2, \dots, A_n are pairwise partially synchronized. Then for any nonnegative numbers $u_1, v_1, u_2, v_2, \dots, u_n, v_n$, we have $\sum_{i=1}^n u_i A_i \sim_p \sum_{i=1}^n v_i A_i$.*

Proof. Since scalars preserve the weak synchronicity relation, we see that the $2n$ sequences $u_i A_i$ and $v_i A_i$ are pairwise partially synchronized. By iterative application, it suffices to show that summation preserves partial synchronicity. Namely, given $A, B, C \in \mathcal{L}^*$, we only need to show that $A + B \sim_p C$ if $A \sim_p C$ and $B \sim_p C$.

Let m and n be integers such that $m \geq n$. The condition $A \sim_p C$ implies that

$$(3.28) \quad a_m c_n + a_n c_m \geq a_{m+1} c_{n-1} + a_{n-1} c_{m+1}.$$

The condition $B \sim_p C$ implies that

$$(3.29) \quad b_m c_n + b_n c_m \geq b_{m+1} c_{n-1} + b_{n-1} c_{m+1}.$$

Adding Ineqs. (3.28) and (3.29) up, one obtains that

$$(a_m + b_m) c_n + (a_n + b_n) c_m \geq (a_{m+1} + b_{m+1}) c_{n-1} + (a_{n-1} + b_{n-1}) c_{m+1}.$$

On the other hand, the sequence $A + B$ is log-concave by Lemma 3.5. Hence, we find $A + B \sim_p C$. This completes the proof. \square

In [5, Theorem 2.12], Gross et al. also showed that the synchronicity relation is preserved by the sequence convolution operation. Example 2.3 illustrates that this property does not hold for weak synchronicity. Below we demonstrate that the same property holds for partial synchronicity.

Theorem 3.7. *Let $A, B, C \in \mathcal{L}^*$. If $A \sim_p B$, then $A * C \sim_p B * C$.*

Proof. Suppose that $A = (a_k)$, $B = (b_k)$, and $C = (c_k)$. Since all the sequences A , B and C are log-concave without internal zeros, so are the sequences $A * C$ and $B * C$. Let m and n be integers such that $m \geq n$. From Definition 3.2 of weakly synchronization, it suffices to show that

$$f(A * C, B * C; m, n) \geq f(A * C, B * C; m + 1, n - 1).$$

that is,

$$(3.30) \quad (A * C)_m (B * C)_n + (A * C)_n (B * C)_m \geq (A * C)_{m+1} (B * C)_{n-1} + (A * C)_{n-1} (B * C)_{m+1},$$

where the notation S_n for a sequence S denotes the n th term of S .

We consider each summand in Ineq. (3.30) as a linear combination of the products of form $c_k c_l$, where $k < l$ are integers. Then the coefficient of $c_k c_l$ in the expansion of the first summand $(A * C)_m (B * C)_n$ is

$$a_{m-k} b_{n-l} + a_{m-l} b_{n-k}.$$

Dealing with the second summand $(A * C)_n (B * C)_m$ by exchanging the numbers m and n in the above expression, we find the coefficient of $c_k c_l$ of the left hand side of Ineq. (3.30) is

$$(3.31) \quad (a_{m-k} b_{n-l} + a_{m-l} b_{n-k}) + (a_{n-k} b_{m-l} + a_{n-l} b_{m-k}) = f(m - k, n - l) + f(m - l, n - k).$$

In Eq. (3.31), replacing m by $m + 1$, and replacing n by $n - 1$, we find that the coefficient of $c_k c_l$ of the right hand side of Ineq. (3.30) is

$$f(m + 1 - k, n - 1 - l) + f(m + 1 - l, n - 1 - k).$$

In the same way, one may check that the coefficients of c_k^2 in the two sides of Ineq. (3.30) are respectively

$$f(m-k, n-k) \quad \text{and} \quad f(m+1-k, n-1-k).$$

To sum up, we can recast the desired Ineq. (3.30) in terms of the function f as

$$(3.32) \quad \begin{aligned} & \sum_{k < l} [f(m-k, n-l) + f(m-l, n-k)] \cdot c_k c_l \\ & - \sum_{k < l} [f(m+1-k, n-1-l) + f(m+1-l, n-1-k)] \cdot c_k c_l \\ & + \sum_k [f(m-k, n-k) - f(m+1-k, n-1-k)] \cdot c_k^2 \geq 0, \end{aligned}$$

where the indices of every summation run over, in fact, a finite number of integers (since the number of non-zero terms in the sequence C is finite). We omit the range of such indices, and adopt this simplicity convention throughout this paper.

We define

$$(3.33) \quad g(k, l) = f(m-k, n-l) - f(m+1-l, n-1-k).$$

Then the desired Ineq. (3.32) can be written simply as

$$\sum_{k < l} [g(k, l) + g(l, k)] c_k c_l + \sum_k g(k, k) c_k^2 \geq 0,$$

that is,

$$(3.34) \quad \sum_{k, l} g(k, l) c_k c_l \geq 0.$$

Let s be a nonnegative integer, indicating the sum $k + l$ of the indices. For notation simplicity, we define

$$(3.35) \quad h_e(k) = g(k, 2s-k),$$

$$(3.36) \quad h_o(k) = g(k, 2s+1-k),$$

where the subscript letter “e” indicates that the sum $2s$ of the two variates k and $2s-k$ in Def. (3.35) is an even integer, and the subscript letter “o” indicates “odd”. By virtue of these notation, the desired Ineq. (3.34) can be recast as

$$\sum_{s, k} [h_e(k) c_k c_{2s-k} + h_o(k) c_k c_{2s+1-k}] \geq 0.$$

Thus, it suffices to show, for all integers $s \geq 0$, that

$$(3.37) \quad \sum_k h_e(k) c_k c_{2s-k} \geq 0$$

and that

$$(3.38) \quad \sum_k h_o(k) c_k c_{2s+1-k} \geq 0.$$

We shall show them individually. Let $s \geq 0$.

We transform the left hand sides of the desired Ineq. (3.37) as

$$(3.39) \quad \sum_k h_e(k) c_k c_{2s-k} = \sum_{k=0}^s (c_{s-k} c_{s+k} - c_{s-k-1} c_{s+k+1}) \sum_{i=s-k-1}^{s+k+1} h_e(i).$$

Since the sequence C is log-concave, we infer that

$$c_{s-k} c_{s+k} - c_{s-k-1} c_{s+k+1} \geq 0.$$

Thus, in view of Eq. (3.39), the desired Ineq. (3.37) holds if

$$\sum_{i=s-k-1}^{s+k+1} h_e(i) \geq 0 \quad \text{for all } 0 \leq k \leq s.$$

Now, let us reduce the left hand side $\sum_{i=s-k-1}^{s+k+1} h_e(i)$. From Def. (3.33) of the function g , it is straightforward to verify that

$$(3.40) \quad g(k, l) + g(l-1, k+1) = 0.$$

Taking $k = s-i$ and $l = s+i$, Eq. (3.40) becomes

$$(3.41) \quad g(s-i, s+i) + g(s+i-1, s-i+1) = 0.$$

By Def. (3.35) of the function h_e , Eq. (3.41) can be rewritten as

$$(3.42) \quad h_e(s-i) + h_e(s+i-1) = 0.$$

By using Eq. (3.42), we can simplify

$$\sum_{i=s-k-1}^{s+k+1} h_e(i) = h_e(s+k+1).$$

Thereby to confirm the desired Ineq. (3.37), it suffices to show $h_e(s+k+1) \geq 0$, that is, $g(s+k+1, s-k-1) \geq 0$. To do this, we will prove a stronger result that

$$(3.43) \quad g(k, l) \geq 0 \quad \text{for all } k \geq l.$$

On the way using Lemma 3.1, one needs to check three conditions. First, the sequences A and B are partially synchronized as in the premise. Second, the sum of variates of the functions

$$f(m-k, n-l) \quad \text{and} \quad f(m+1-l, n-1-k)$$

are equal, i.e.,

$$(m-k) + (n-l) = (m+1-l) + (n-1-k).$$

Third, since $m \geq n$ and $k \geq l$, the distances of variates are comparative as

$$|(m - k) - (n - l)| \leq m - n < |(m + 1 - l) - (n - 1 - k)|.$$

By Lemma 3.1, we deduce the claimed Ineq. (3.43).

When $k \geq 0$, we have $s + k + 1 \geq s - k - 1$, and hence by Ineq. (3.43),

$$h_e(s + k + 1) = g(s + k + 1, s - k - 1) \geq 0.$$

This completes the proof of Ineq. (3.37).

Inequality (3.38) can be shown along the same lines. In fact,

$$\sum_k h_o(k) c_k c_{2s+1-k} = \sum_{k=0}^s (c_k c_{2s+1-k} - c_{k-1} c_{2s+2-k}) \sum_{i=k}^{2s+1-k} h_o(i).$$

Since the sequence C is log-concave, we have

$$c_k c_{2s+1-k} - c_{k-1} c_{2s+2-k} \geq 0.$$

Thus the desired Ineq. (3.37) holds if

$$\sum_{i=k}^{2s+1-k} h_o(i) \geq 0 \quad \text{for all } 0 \leq k \leq s.$$

On the other hand, from Def. (3.36) of the function h_o , Eq. (3.40) implies that

$$(3.44) \quad h_o(i) + h_o(2s - i) = 0.$$

In particular, taking $i = s$ in Eq. (3.44), one obtains that

$$(3.45) \quad h_o(s) = 0.$$

By using Eqs. (3.44) and (3.45), we can simplify

$$\sum_{i=k}^{2s+1-k} h_o(i) = h_o(2s + 1 - k).$$

When $0 \leq k \leq s$, we have $2s + 1 - k \geq k$. Hence, Def. (3.36) and Ineq. (3.43) imply that

$$h_o(2s + 1 - k) = g(2s + 1 - k, k) \geq 0.$$

This completes the proof. □

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SCHOOL OF MATHEMATICAL SCIENCES, & LMAM, PEKING UNIVERSITY, 100871 BEIJING, P. R. CHINA; EMAIL: HUHAN@PKU.EDU.CN

[†]SCHOOL OF MATHEMATICS AND STATISTICS, BEIJING INSTITUTE OF TECHNOLOGY, 102488 BEIJING, P. R. CHINA, [‡]BEIJING KEY LABORATORY ON MCAACI, BEIJING INSTITUTE OF TECHNOLOGY, 102488 BEIJING, P. R. CHINA, EMAIL: GLW@BIT.EDU.CN

SCHOOL OF MATHEMATICAL SCIENCES, & LMAM, PEKING UNIVERSITY, 100871 BEIJING, P. R. CHINA; EMAIL: ZHF327@PKU.EDU.CN

SCHOOL OF MATHEMATICAL SCIENCES, & LMAM, PEKING UNIVERSITY, 100871 BEIJING, P. R. CHINA; EMAIL: ZHAOTONGYUAN@PKU.EDU.CN